

$N = 1$ supersymmetry and the three loop anomalous dimension for the chiral superfield

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We calculate the three loop anomalous dimension for a general $N = 1$ supersymmetric gauge theory. The result is used to probe the possible existence of renormalisation invariant relationships between the Yukawa and gauge couplings.

1. Introduction

In a recent paper [1] we showed that in certain supersymmetric theories, it is possible to impose a relation among the dimensionless couplings (which we shall call generically the $P = \frac{1}{3}Q$ condition), which is preserved under renormalisation through at least two loops. In these theories, then if $P = \frac{1}{3}Q$ is not actually imposed, it nevertheless represents a renormalisation-group (RG) fixed point of the coupling evolution: in many theories an infra-red stable one. In addition we showed that in such theories the soft supersymmetry-breaking couplings may take (or approach in the infra-red) a particular universal form which is also preserved by RG evolution.

We have pursued the phenomenological consequences of these ideas elsewhere[2]. These are valid whether $P = \frac{1}{3}Q$ is an infra-red phenomenon [3] [4] or a consequence of some fundamental symmetry. In this paper we ask whether the special properties of theories with $P = \frac{1}{3}Q$ persist to higher orders. We postulate an all-orders relation between the gauge β -function and the anomalous dimension of the chiral supermultiplet for a $P = \frac{1}{3}Q$ theory, and then investigate its validity. For this purpose it is sufficient to calculate the three-loop contribution $\gamma^{(3)}$ to the anomalous dimension of the chiral supermultiplet in a general $N = 1$ supersymmetric gauge theory. This calculation is an extension to the existing one of Parkes[5], where he calculated $\gamma^{(3)}$ for a non-abelian one-loop finite theory, in turn generalising the existing result of Refs. [6] and [7] for the $N = 4$ theory.

Our result will have other uses; we may, for example, calculate the Yukawa β -functions for the supersymmetric standard model and investigate the domain of perturbative believability for the t -quark Yukawa coupling.

2. The $P = \frac{1}{3}Q$ condition

The Lagrangian $L_{\text{SUSY}}(W)$ for a $N = 1$ supersymmetric theory is defined by the superpotential

$$W = \frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{2}\mu^{ij}\Phi_i\Phi_j. \quad (2.1)$$

L_{SUSY} is the Lagrangian for the $N = 1$ supersymmetric gauge theory, containing the gauge multiplet (λ being the gaugino) and a chiral superfield Φ_i with component fields $\{\phi_i, \psi_i\}$ transforming as a representation R of the gauge group \mathcal{G} . We assume that there are no gauge-singlet fields and that \mathcal{G} is simple.

The superpotential W undergoes no infinite renormalisation; so that we have, using minimal subtraction (MS) or modified minimal subtraction ($\overline{\text{MS}}$)

$$\beta_Y^{ijk} = Y^{p(ij}\gamma^k)_{p} = Y^{ijp}\gamma^k_{p} + (k \leftrightarrow i) + (k \leftrightarrow j), \quad (2.2)$$

where γ is the anomalous dimension for Φ . Note that Eq. (2.2) depends on use of MS or $\overline{\text{MS}}$ as well as on a supersymmetric regularisation method; we shall see later that this relationship between β_Y and γ is not preserved by certain coupling constant redefinitions.

The one-loop results for the gauge coupling β -function β_g and for γ are given by

$$16\pi^2\beta_g^{(1)} = g^3Q, \quad \text{and} \quad 16\pi^2\gamma^{(1)i}_{j} = P^i_{j}, \quad (2.3)$$

where

$$Q = T(R) - 3C(G), \quad \text{and} \quad (2.4a)$$

$$P^i_{j} = \frac{1}{2}Y^{ikl}Y_{jkl} - 2g^2C(R)^i_{j}. \quad (2.4b)$$

Here

$$T(R)\delta_{AB} = \text{Tr}(R_AR_B), \quad C(G)\delta_{AB} = f_{ACD}f_{BCD} \quad \text{and} \quad C(R)^i_{j} = (R_AR_A)^i_{j}. \quad (2.5)$$

The two-loop β -functions for the dimensionless couplings were calculated in Refs. [8]–[12]:

$$(16\pi^2)^2\beta_g^{(2)} = 2g^5C(G)Q - 2g^3r^{-1}C(R)^i_{j}P^j_{i} \quad (2.6a)$$

$$(16\pi^2)^2\gamma^{(2)i}_{j} = [-Y_{jmn}Y^{mpi} - 2g^2C(R)^p_{j}\delta^i_{n}]P^n_{p} + 2g^4C(R)^i_{j}Q, \quad (2.6b)$$

where Q and P^i_{j} are given by Eq. (2.4), and $r = \delta_{AA}$.

The $P = \frac{1}{3}Q$ condition is the requirement that

$$P^i_{j} = \frac{1}{3}g^2Q\delta^i_{j}, \quad (2.7)$$

or equivalently

$$\gamma^{(1)} = \frac{\beta_g^{(1)}}{3g}\delta^i_{j}, \quad (2.8)$$

and thus amounts to a postulated relation between the Yukawa and gauge couplings. It is easy to show from Eqs. (2.2)–(2.4) and (2.6) that Eq. (2.7) corresponds to a fixed point

in the evolution of Y^{ijk}/g , up to two-loop order; in other words, a possible solution to the equation

$$\mu \frac{d}{d\mu} \frac{Y^{ijk}}{g} = 0, \quad (2.9)$$

or equivalently

$$\beta_Y^{ijk} = g^{-1} Y^{ijk} \beta_g. \quad (2.10)$$

While Eq. (2.7) solves Eq. (2.10) up to two loops, it is highly restrictive, and in many cases there is no choice of the Yukawa couplings that corresponds to a solution of Eq. (2.7). The t -quark Yukawa evolution in the standard model (or indeed in the supersymmetric standard model) is a familiar example of this type [13]. The attractive feature of theories that do admit $P = \frac{1}{3}Q$ is that in such theories, the soft supersymmetry-breaking couplings also have fixed points which correspond to the commonly assumed universal form[1]–[3].

In general, the solution to Eq. (2.10) (to all orders) would be a power series of the form

$$Y^{ijk} = g a_1^{ijk} + g^3 a_3^{ijk} + g^5 a_5^{ijk} + \dots, \quad (2.11)$$

where a_1 , a_3 , a_5 etc are constant tensors. What we have shown is that $a_3^{ijk} = 0$ if $Y^{ijk} = g a_1^{ijk}$ satisfies Eq. (2.7), in other words if

$$\frac{1}{2} a_1^{ikl} a_{1jkl} - 2C(R)^i_j = \frac{1}{3} Q \delta^i_j. \quad (2.12)$$

Our investigation in this paper is motivated by the question of whether Eq. (2.7) corresponds to a fixed point of Y^{ijk}/g to all orders, or in other words whether in Eq. (2.11), we have $a_5 = a_7 = \dots = 0$ in addition to $a_3 = 0$, so that

$$Y^{ijk} = g a_1^{ijk} \quad (2.13)$$

gives a fixed point of Y^{ijk}/g to all orders.

In a situation of physical interest such as a supersymmetric grand unified theory, the symmetry of some underlying theory might guarantee that Eq. (2.7) was satisfied at energies of order the Planck scale. If Eq. (2.7) corresponded to an exact fixed point, then it would continue to be exactly satisfied during the RG evolution. On the other hand, the fixed point might be approached in the infra-red in the absence of any special boundary conditions at the Planck scale.

Fixed points of Y^{ijk}/g are related to the coupling constant reduction (CCR) program pursued in Ref. [14]. A fixed point of the form Eq. (2.13) is the simplest possible realisation

of CCR. In general, CCR proceeds by the assumption that in a many-coupling theory, all the couplings may be expressed in terms of one of them (typically the gauge coupling) by relations which in our notation would take the form:

$$Y^{ijk} = \lambda^{ijk}(g) \quad (2.14)$$

whence

$$\beta_Y^{ijk} = \frac{d\lambda^{ijk}}{dg} \beta_g. \quad (2.15)$$

Clearly Eq. (2.13) corresponds to the simplest possible result for λ^{ijk} ; in general one might expect

$$\lambda^{ijk} = ga^{ijk} + g^3 b^{ijk} + g^5 c^{ijk} + \dots, \quad (2.16)$$

where a^{ijk} , b^{ijk} , c^{ijk} etc are μ -independent constant tensors. We shall see later that this general situation is equivalent to a fixed point for a redefined Y^{ijk}/g . We shall return to a discussion of CCR in the $P = \frac{1}{3}Q$ case, for which $a^{ijk} = a_1^{ijk}$, later.

The $P = \frac{1}{3}Q$ condition itself is RG invariant at a fixed point of Y^{ijk}/g . Indeed, differentiating Eq. (2.7) with respect to μ we obtain

$$\frac{1}{2} \{ \beta_Y^{ikl} Y_{jkl} + Y^{ikl} \beta_{Y_{jkl}} \} - 4g\beta_g C(R)^i_j = \frac{2}{3}g\beta_g Q \delta^i_j, \quad (2.17)$$

which is satisfied when Eq. (2.10) holds; in other words Eq. (2.10) implies that the $P = \frac{1}{3}Q$ condition is RG invariant. It is easy to check that the reason Eq. (2.10) is satisfied up to two loops by couplings satisfying Eq. (2.7) is that Eq. (2.7) also implies

$$\gamma^{(2)i}_j = \frac{\beta_g^{(2)}}{3g} \delta^i_j, \quad (2.18)$$

which readily follows from Eqs. (2.6) and (2.4). This corresponds to having $b^{ijk} = 0$ in Eq. (2.16). It is then natural to speculate that this relation might be completely general, so that

$$\gamma^i_j = \frac{\beta_g}{3g} \delta^i_j \quad (2.19)$$

to all orders, provided we impose Eq. (2.7). It is this hypothesis which we aim to check.

At this point we encounter scheme-dependence problems. It is obvious without any calculations that Eq. (2.19) will not be true if we use DRED and $\overline{\text{MS}}$ or $\overline{\text{MS}}$, since we know [5][10] that in a two-loop finite theory, $\gamma^{(3)}$ is non-zero, while $\beta_g^{(3)}$ is zero. All is not lost, however, since in such a case $\gamma^{(3)}$ may be transformed to zero by a coupling constant

redefinition (as we shall show later). Such redefinitions are equivalent to a change of renormalisation scheme. We might have hoped, therefore, that in this new scheme Eq. (2.19) would hold in the non-finite case. We will see, however, that (essentially because $C(R)$ is not in general proportional to the unit matrix) it is not possible to achieve Eq. (2.19) even with arbitrary coupling constant redefinitions. Even had we succeeded, however, the significance of the result for the fixed point discussion would have been unclear, since after coupling constant redefinitions Eq. (2.2) no longer holds (except in the finite case), so that Eq. (2.19) no longer corresponds to a solution of Eq. (2.10). We discuss these issues in more detail in section 7.

3. γ and the Instanton-based β_g

There exists an exact relation between γ and β_g , derived by Novikov et al. and based on the instanton calculus [15][16]. In our notation it reads:

$$\beta_g = \frac{g^3}{16\pi^2} \left[\frac{Q - 2r^{-1}\text{Tr}[\gamma C(R)]}{1 - 2C(G)g^2(16\pi^2)^{-1}} \right]. \quad (3.1)$$

It is not entirely clear from Ref. [15], [16] how general this result for β_g is. For example, in Ref. [16], one sees from the expression given for $\gamma^{(1)}$ that in this paper $Y^{ijk} = 0$ is assumed. In Ref. [15], however, it is clearly asserted that the result stands for non-zero Yukawa couplings. Also applications of Eq. (3.1) of which we are aware have been to cases such that $C(R) \sim \delta^i_j$. We shall in any case assume that Eq. (3.1) is true for a general theory with a superpotential.

In the case $\gamma = 0$ (no matter fields) Eq. (3.1) was first obtained in Ref. [17]; see also Ref. [10] and Ref. [18]. Note that in Eq. (3.1), the $(n+1)$ th loop contribution to β_g , $\beta_g^{(n+1)}$, is essentially determined by $\gamma^{(n)}$, and that for an n -loop finite theory we have automatically that $\beta_g^{(n+1)} = 0$. Thus Eq. (3.1) is consistent with Refs. [10] and [19] where the same result is advocated. We can use the result for $\gamma^{(2)}$, Eq. (2.6b), to find $\beta_g^{(3)}$:

$$\begin{aligned} (16\pi^2)^3 \beta_g^{(3)} = & 4g^7 Q C(G)^2 - 4g^5 C(G) r^{-1} 16\pi^2 \text{Tr} \left[\gamma^{(1)} C(R) \right] \\ & - 2g^3 r^{-1} (16\pi^2)^2 \text{Tr} \left[\gamma^{(2)} C(R) \right]. \end{aligned} \quad (3.2)$$

Later we will use our result for $\gamma^{(3)}$ to deduce $\beta_g^{(4)}$.

It follows from Eqs. (3.2), (2.3), (2.4) and (2.6b) that in the $P = \frac{1}{3}Q$ case we have

$$\beta_g^{(3)} = \frac{4}{9} \frac{g^7 Q^3}{(16\pi^2)^3}. \quad (3.3)$$

Our hypothesis is that a scheme exists in which our result for $\gamma^{(3)}$ satisfies Eq. (2.19), and so Eq. (3.3) then implies

$$\gamma^{(3)} = \frac{4}{27} \frac{g^6 Q^3}{(16\pi^2)^3}. \quad (3.4)$$

We should emphasise that $\beta_g^{(3)}$ is sensitive to coupling constant redefinitions of the form $\delta g \sim O(g^5), (g^3 Y^2)$ etc, but *not* to redefinitions δY of the same order: such redefinitions of Y we will employ later.

There is little doubt that we should use the MS results at one and two loops on the right-hand side of Eq. (3.2). However, at higher order, it is not immediately clear in which scheme Eq. (3.1) will be valid. The obvious expectation would be that MS is appropriate. On the other hand, it is claimed [20] that there are schemes in which a one-loop finite theory is finite to all orders; clearly such a scheme would not correspond to MS or $\overline{\text{MS}}$. Since as we already mentioned, Eq. (3.1) gives $\beta_g^{(n+1)} = 0$ for a n -loop finite theory, it might be natural to believe that it would be a scheme of this type in which Eq. (3.1) holds. From our point of view the best-case scenario would be that there was a scheme in which Eq. (2.19) and Eq. (3.1) both held to all orders. We could then solve for both β_g and γ , obtaining

$$\gamma^i_j = \frac{1}{3g} \beta_g \delta^i_j = \frac{g^2}{16\pi^2} \left[\frac{Q}{3 + 2Qg^2(16\pi^2)^{-1}} \right] \delta^i_j. \quad (3.5)$$

We then see that such a scheme would also be one for which a one-loop finite theory is all-orders finite.

4. Three-loop finiteness

In this section we review Parkes's result for $\gamma^{(3)}$ in a general one-loop finite theory, and show how a coupling constant redefinition reveals a finite theory. Thus throughout this section we shall set $P = Q = 0$. The result for $\gamma^{(3)}$ in this special case, we will call $\gamma_P^{(3)}$, is given by: †

$$\begin{aligned} (16\pi^2)^3 \gamma_P^{(3)} = & \kappa g^6 [12C(R)C(G)^2 - 2C(R)^2 C(G) - 10C(R)^3 - 4C(R)\Delta(R)] \\ & + \kappa g^4 [4C(R)S_1 - C(G)S_1 + S_2 - 5S_3] + \kappa g^2 Y^* S_1 Y + \kappa M/4 \end{aligned} \quad (4.1)$$

† We use the usual particle physicist's definition of the gauge coupling g ; to compare with Ref. [5] one must set $g = 1/\sqrt{2}$

where

$$S_{1j}^i = Y^{imn} C(R)^p{}_m Y_{jpn} \quad (4.2a)$$

$$(Y^* S_1 Y)^i{}_j = Y^{imn} S_1^p{}_m Y_{jpn} \quad (4.2b)$$

$$S_{2j}^i = Y^{imn} C(R)^p{}_m C(R)^q{}_n Y_{jpq} \quad (4.2c)$$

$$S_{3j}^i = Y^{imn} (C(R)^2)^p{}_m Y_{jpn} \quad (4.2d)$$

$$M^i{}_j = Y^{ikl} Y_{kmn} Y_{lrs} Y^{pmr} Y^{qns} Y_{jpq} \quad (4.2e)$$

$$\Delta(R) = \sum_{\alpha} C(R_{\alpha}) T(R_{\alpha}) \quad (4.2f)$$

and $\kappa = 6\zeta(3)$. In Eq. (4.2f) the sum over α is a sum over irreducible representations. Thus whereas $C(R)$ is a matrix, $C(R_{\alpha})$ and $\Delta(R)$ are numbers.

Now let us consider the effect on γ_P of coupling constant redefinitions. Under a coupling constant redefinition of the form

$$Y^{ijk} \rightarrow Y^{ijk} + \delta Y^{ijk} \quad \text{and} \quad g \rightarrow g + \delta g \quad (4.3)$$

we have

$$(\delta\gamma_P)^i{}_j = -\frac{1}{2}(Y^{ikl}\delta Y_{jkl} + \delta Y^{ikl}Y_{jkl} - 8gC(R)^i{}_j\delta g) \quad (4.4)$$

We can exploit the freedom to make such coupling redefinitions to change $\gamma_P^{(3)}$ so that it vanishes in a one-loop finite theory, as follows. Let us choose

$$\begin{aligned} (16\pi^2)^3 \delta Y^{ijk} = & k_1 g^2 S_1^{(i}{}_m Y^{jk)m} + g^4 [k_2 C(R)^{(i}{}_m C(R)^{j)}{}_n Y^{k)mn} \\ & + k_3 Y^{n(jk} C(R)^{i)}{}_m C(R)^m{}_n + k_4 \Delta(R) Y^{ijk} \\ & + k_5 C(G) C(R)^{(i}{}_m Y^{jk)m} + k_6 C(G)^2 Y^{ijk}] \\ & + k_7 Y^{ilm} Y^{jpq} Y^{krs} Y_{lpr} Y_{mqs} \end{aligned} \quad (4.5)$$

(Note that our notation for symmetrising over (ijk) involves a sum over three terms only: see Eq. (2.2).) The result is

$$\begin{aligned} (16\pi^2)^3 \delta\gamma_P = & -2k_1 g^2 (Y^* S_1 Y + 2g^2 C(R) S_1 + P S_1) - k_2 g^4 (S_2 + 2C(R) S_1) \\ & - 2k_3 g^4 (S_3 + 2g^2 C(R)^3 + P C(R)^2) - k_4 g^4 \Delta(R) [2P + 4g^2 C(R)] \\ & - 2k_5 g^4 C(G) [S_1 + 2g^2 C(R)^2 + P C(R)] - k_6 g^4 C(G)^2 [2P + 4g^2 C(R)] \\ & - k_7 M, \end{aligned} \quad (4.6)$$

where, for later reference, we have temporarily reinstated factors of P . Then setting $P = 0$ and

$$\begin{aligned} k_1 &= \frac{1}{2}\kappa, & k_2 &= \kappa, & k_3 &= -\frac{5}{2}\kappa, & k_4 &= -\kappa, \\ k_5 &= -\frac{1}{2}\kappa, & k_6 &= 3\kappa, & k_7 &= \frac{1}{4}\kappa \end{aligned} \quad (4.7)$$

we obtain $\delta\gamma_P = -\gamma_P^{(3)}$. Thus in an arbitrary one-loop finite theory one can transform $\gamma_P^{(3)}$ to zero. This simple demonstration from Parkes's result seems to have eluded some previous authors [†] who have verified that it can be done in specific cases. There exist arguments that this procedure may in fact be extended to all orders[20]; a specific calculation is always comforting, however, and in fact in some of the arguments presented in Refs. [20] it is not entirely clear whether there is any constraint necessary with regard to the number of fields vis-à-vis the number of independent couplings. At least at three loops, we now see that there is none. (It is important that by one-loop finiteness we mean $\gamma^{(1)} = \beta_g^{(1)} = 0$; it is easy to construct theories such that $\beta_Y^{(1)} = \beta_g^{(1)} = 0$, which are not even two-loop finite, and cannot be rendered so by coupling constant redefinitions.)

We should note that for the finite theory, the effect on $\gamma^{(3)}$ of a redefinition $\delta g = ag$ is the same as that of the redefinition $\delta Y^{ijk} = -aY^{ijk}$ (on use of the condition $P = 0$), and hence we could replace the k_4 and k_6 terms in Eq. (4.5) with a transformation on g of the form

$$\delta g = g^4 [k_4 \Delta(R) + k_6 C(G)^2]. \quad (4.8)$$

This would, however, change $\beta_g^{(3)}$; in particular we would no longer have $\beta_g^{(3)} = 0$ in $N = 2$ theories (except, of course, for one-loop finite ones). We will return to the special case of $N = 2$ in section 6.

We have concentrated above on the effect of coupling constant redefinitions on γ_P ; what is the effect of the corresponding redefinition on the β -function β_Y ? In general we have to leading order in $\delta Y, \delta g$ that

$$\delta\beta_Y = \left[\beta_Y \cdot \frac{\partial}{\partial Y} + \beta_Y^* \cdot \frac{\partial}{\partial Y^*} + \beta_g \cdot \frac{\partial}{\partial g} \right] \delta Y - \left[\delta Y \cdot \frac{\partial}{\partial Y} + \delta Y^* \cdot \frac{\partial}{\partial Y^*} + \delta g \cdot \frac{\partial}{\partial g} \right] \beta_Y \quad (4.9)$$

with a similar formula for $\delta\beta_g$. It is easy to see from this result that in a one loop finite theory we have simply

$$\delta\beta_Y = Y^{m(ij} \delta\gamma_P^{k)}{}_{m}, \quad (4.10)$$

[†] Including, it must be said, one of the present authors[21]

since at one loop we have $\beta_Y = \beta_g = 0$. Hence if $\gamma_P^{(3)} + \delta\gamma_P = 0$ then $\beta_Y^{(3)} + \delta\beta_Y = 0$ likewise. It should be clear, however, that if we do not have $P = Q = 0$, the situation changes and that after a coupling constant redefinition the new β_Y and the new γ are not necessarily related by Eq. (2.2). We will return to this point in section 7.

5. $\gamma^{(3)}$ for a general theory

Here we present the result for $\gamma^{(3)}$ for a general non-abelian theory. The calculation is a straightforward application of the superfield Feynman rules spelled out in Ref. [22] and applied to complementary calculations in Refs. [5], [6] and [7]. (See also Ref. [11], where $\gamma^{(2)}$ is calculated for a general $N = 1$ theory). In the $N = 4$ case the set of Feynman diagrams to be calculated are to be found in Fig. 5 and Fig. 6 of Ref. [6]; we have to add diagrams with one or more one-loop self-energy insertions, and also a set of non-planar diagrams which happen to have vanishing group theory factors in $N = 4$ (Parkes calculated the latter graphs too).

One way in which we differ somewhat from some early calculations is that we have performed the calculation in the Feynman gauge. In the literature one finds statements to the effect that it is more convenient or even essential to restore the radiatively corrected gauge boson propagator to Feynman gauge form by redefining the gauge parameter. If this is done then the seagull graphs of the type shown in Fig. 7(b) of Ref. [6] are zero; for us, however these are non-zero, but the graphs of the type Fig. 6(c) of the same reference are zero instead, because of the transverse nature of the gauge boson self energy. We have in fact checked that both procedures lead to the same result, but it seems to us simpler to stick to the Feynman gauge. We evaluate the Feynman integrals by setting the external momentum zero, and introducing masses as necessary to control infra-red divergences. We then perform subtractions at the level of the Feynman integrals. (A similar procedure is described in Ref. [19].) This procedure gives an unambiguous result. No explicit factors of d arise in the algebra; this is important since such d -dependence would require careful handling. (For a discussion, see for example Ref. [23]). As explained in more detail later, we did not use the special case of $N = 2$ supersymmetry except as a final check, but we did use some of the graph-by-graph results from Ref. [6] for $N = 4$ to avoid particularly tedious calculations.

The result for $\gamma^{(3)}$ in a general theory is:

$$\begin{aligned}
(16\pi^2)^3 \gamma^{(3)} = & (16\pi^2)^3 \gamma_P^{(3)} \\
& + \kappa \{ g^2 [C(R)S_4 - 2S_5 - S_6] - g^4 [PC(R)C(G) + 5PC(R)^2] \\
& + 4g^6 QC(G)C(R) \} + 2Y^* S_4 Y - \frac{1}{2} S_7 - S_8 + g^2 [4C(R)S_4 + 4S_5] \quad (5.1) \\
& + g^4 [8C(R)^2 P - 2QC(R)P - 4QS_1 - 10r^{-1} \text{Tr} [PC(R)] C(R)] \\
& + g^6 [2Q^2 C(R) - 8C(R)^2 Q + 10QC(R)C(G)]
\end{aligned}$$

where

$$S_{4j}^i = Y^{imn} P_m^p Y_{jpn} \quad (5.2a)$$

$$S_{5j}^i = Y^{imn} C(R)^p_m P_p^q Y_{j n q} \quad (5.2b)$$

$$S_{6j}^i = Y^{imn} C(R)^p_m P_n^q Y_{j p q} \quad (5.2c)$$

$$S_{7j}^i = Y^{imn} P_m^p P_n^q Y_{j p q} \quad (5.2d)$$

$$S_{8j}^i = Y^{imn} (P^2)^p_m Y_{j p n} \quad (5.2e)$$

$$(Y^* S_4 Y)^i_j = Y^{imn} S_4^p_m Y_{j p n}. \quad (5.2f)$$

Eq. (5.1) is our main result. It is easy to see that for $P = Q = 0$ it reduces to Parkes's result, Eq. (4.1).

We can use this result for $\gamma^{(3)}$ to write down $\beta_g^{(4)}$:

$$\begin{aligned}
(16\pi^2)^4 \beta_g^{(4)} = & 8g^9 QC(G)^3 - 8g^7 C(G)^2 r^{-1} 16\pi^2 \text{Tr} [\gamma^{(1)} C(R)] \\
& - 4C(G) g^5 r^{-1} (16\pi^2)^2 \text{Tr} [\gamma^{(2)} C(R)] - 2g^3 r^{-1} (16\pi^2)^3 \text{Tr} [\gamma^{(3)} C(R)]. \quad (5.3)
\end{aligned}$$

Naturally the precise result for $\beta_g^{(4)}$ depends on the scheme that we use to evaluate $\gamma^{(3)}$; as we already indicated, it is probably appropriate to use MS, that is Eq. (5.1).

Let us compare our result for $\gamma^{(3)}$ with previous calculations. The simplest possible case is the Wess-Zumino model, corresponding to $g = 0$ and a superpotential $W = \frac{1}{6} \lambda \Phi^3$. There are several results to choose from in the literature; the original calculation [24], and two subsequent efforts [25] [26], both of which in fact proceeded to four loops. The three calculations differ with regard to the coefficient of the $\zeta(3)$ term at three loops.

It is easy to show that for the Wess-Zumino model our result is

$$\gamma = \frac{1}{2} \left(\frac{\lambda}{4\pi} \right)^2 - \frac{1}{2} \left(\frac{\lambda}{4\pi} \right)^4 + \left(\frac{5}{8} + \frac{3}{2} \zeta(3) \right) \left(\frac{\lambda}{4\pi} \right)^6 + \dots \quad (5.4)$$

and that this agrees with Ref. [26].

Another interesting check on our result is afforded by the case of $N = 2$ supersymmetry. We discuss this in the next section and then consider the effect of coupling constant redefinitions on Eq. (5.1), with emphasis on theories that satisfy $P = \frac{1}{3} Q$.

6. The $N = 2$ case

In $N = 1$ language, an $N = 2$ theory is defined by the superpotential

$$W = \sqrt{2}g\eta^a\chi^i S^{aj}{}_i\xi_j \quad (6.1)$$

where η, χ and ξ transform according to the adjoint, S^* and S representations respectively. The set of chiral superfields χ, ξ is called a hypermultiplet. (In the abelian case η is neutral and χ, ξ are two sets of fields in opposite charge pairs. Abelian $N = 2$ without hypermultiplets is a free field theory; but with them we have non-trivial interactions). $N = 2$ theories have one loop divergences only[27]; using $\overline{\text{MS}}$ or $\overline{\text{MS}}$ we may therefore expect that the anomalous dimension of both the η and the hypermultiplet should vanish beyond one loop. Parkes, in fact, used this result to reduce calculational labour; we have preferred to use it as a check.

At the one-loop level we have

$$\begin{aligned} Q &= 2[T(S) - C(G)] \\ P_\eta &= Q\delta^a{}_b \\ P_\chi &= P_\xi = 0. \end{aligned} \quad (6.2)$$

Thus except for the case $P = Q = 0$, $N = 2$ theories cannot satisfy $P = \frac{1}{3}Q$.

In Table 1 we give expressions for group theory factors defined in Eqs. (4.2) and (5.2), when specialised to $N = 2$; using these results we may readily demonstrate that Eq. (5.1) vanishes identically for both η and the hypermultiplet. We note also in passing that the exact β function result in Eq. (3.1) is consistent with neither β_g nor γ receiving corrections beyond one loop in the $N = 2$ case.

Group theory factor	Contribution to η	Contribution to ξ, χ
$C(R)S_1$	$4C(G)\Delta(S)$	$2C(S)^2 [C(S) + C(G)]$
$C(G)S_1$	$4C(G)\Delta(S)$	$2C(S)C(G) [C(S) + C(G)]$
QS_1	$4Q\Delta(S)$	$2QC(S) [C(S) + C(G)]$
Y^*S_1Y	$8C(G)\Delta(S)$ $+8\sum [C(S_\alpha)]^2 T(S_\alpha)$	$4 [C(S) + C(G)] C(S)^2$ $+8C(S)\Delta(S)$
S_2	$4\sum [C(S_\alpha)]^2 T(S_\alpha)$	$4C(G)C(S)^2$
S_3	$4\sum [C(S_\alpha)]^2 T(S_\alpha)$	$2 [C(S)^2 + C(G)^2] C(S)$
$C(R)S_4$	0	$2QC(S)^2$
Y^*S_4Y	$8Q\Delta(S)$	$4QC(S)^2$
S_5	0	$2QC(G)C(S)$
S_6	0	$2QC(S)^2$
S_7	0	0
S_8	0	$2Q^2C(S)$
$C(R)\Delta(R)$	$C(G) [C(G)^2 + 2\Delta(S)]$	$C(S) [C(G)^2 + 2C(S)\Delta(S)]$
M	$16C(G) [C(G)T(S) - 3\Delta(S)]$ $+32\sum C(S_\alpha)^2 T(S_\alpha)$	$16[C(G)^2 - 3C(S)C(G)$ $+2C(S)^2]C(S)$
$C(R)^2P$	$C(G)^2Q$	0
$C(R)^2Q$	$C(G)^2Q$	$C(S)^2Q$
$QC(R)P$	$Q^2C(G)$	0
$Q^2C(R)$	$Q^2C(G)$	$Q^2C(S)$
$\text{Tr}[PC(R)]C(R)$	$QC(G)^2$	$QC(G)C(S)$

Table 1: Group theory factors for the special case of $N = 2$ supersymmetry

7. Coupling constant redefinitions and $\mathbf{P} = \frac{1}{3}\mathbf{Q}$

We now wish to investigate whether our three-loop result for $\gamma^{(3)}$ can be transformed by coupling constant redefinition into the conjectured formula Eq. (3.4) in the case where we impose Eq. (2.7), in accord with our hypothesis. Clearly the redefinition must at least include the redefinition of Eq. (4.6) in order to remove the terms in $\gamma_P^{(3)}$. If we make this

redefinition, then we obtain

$$\begin{aligned}
(16\pi^2)^3(\gamma^{(3)} + \delta\gamma_P) = & \kappa \{ g^2 [C(R)S_4 - 2S_5 - S_6 - PS_1] + 2g^4 P [\Delta(R) - 3C(G)^2] \\
& + 4g^6 QC(G)C(R) \} + 2Y^* S_4 Y - \frac{1}{2} S_7 - S_8 \\
& + g^2 [4C(R)S_4 + 4S_5] \\
& + g^4 [8C(R)^2 P - 2QC(R)P - 4QS_1 - 10r^{-1} \text{Tr} [PC(R)] C(R)] \\
& + g^6 [2Q^2 C(R) - 8C(R)^2 Q + 10QC(R)C(G)] ,
\end{aligned} \tag{7.1}$$

where we have not yet imposed Eq. (2.7). Note that we have now included the terms involving P in Eq. (4.6). If we now impose $P = \frac{1}{3}Q$, then the result simplifies dramatically:

$$\begin{aligned}
(16\pi^2)^3(\gamma^{(3)} + \delta\gamma_P) = & \kappa \left\{ -\frac{4}{3}g^4 QS_1 + g^6 \left[\frac{4}{3}QC(R)^2 + 4QC(R)C(G) \right. \right. \\
& + \left. \frac{2}{3}Q\Delta(R) - 2QC(G)^2 + \frac{2}{9}Q^2 C(R) \right] \Big\} \\
& + \frac{5}{27}g^6 Q^3 .
\end{aligned} \tag{7.2}$$

We can now try to redefine $\gamma^{(3)}$ still further in order to remove the terms proportional to κ . Let us focus our attention on the terms involving QS_1 and $QC(R)^2$. There is one coupling constant redefinition which changes the coefficients of both these terms simultaneously, namely

$$(16\pi^2)^3 \delta Y^{ijk} = k_8 g^4 QC(R)^{(i_l} Y^{jk)l}. \tag{7.3}$$

The corresponding change in $\gamma^{(3)}$ is given according to Eq. (4.4) by

$$(16\pi^2)^3 \delta\gamma_Q = -2k_8 Q g^4 \left[S_1 + 2g^2 C(R)^2 + \frac{1}{3}QC(R) \right]. \tag{7.4}$$

There is no other redefinition which affects the coefficients of either QS_1 or $QC(R)^2$. Since the coefficients of QS_1 and $QC(R)^2$ are in a different ratio in Eq. (7.2) compared with Eq. (7.4), we cannot simultaneously redefine both of them to zero. Hence we cannot remove all the terms proportional to κ in $\gamma^{(3)}$, and so our conjectured result Eq. (3.4) cannot be true in any renormalisation scheme. Moreover, the fact that neither QS_1 nor $QC(R)^2$ is in general proportional to the unit matrix means that Eq. (2.19) cannot be true at three loops whatever the value of $\beta_g^{(3)}$. However, in principle, we could remove all the remaining terms proportional to κ by a redefinition of the form

$$\begin{aligned}
(16\pi^2)^3 \delta Y^{ijk} &= a g^4 Y^{ijk}, \\
(16\pi^2)^3 \delta g &= b g^5,
\end{aligned} \tag{7.5}$$

which produces a change in γ of the form

$$(16\pi^2)^3 \delta\gamma = 4(b-a)g^6 C(R) - \frac{2}{3}g^6 aQ. \quad (7.6)$$

We should note, though, that the redefinition of g would in principle be fixed by a three-loop calculation of β_g . A redefinition $\delta g = bg^5$ produces a change $\delta\beta_g = 2bQg^7$ in β_g . After computing β_g at three loops in MS, δg would be fixed by requiring that it transform the result to the form corresponding to Eq. (3.3). (Since $\beta_g^{(1)}$ contains no Y -dependence, redefinitions of Y of the form we are considering in Eqs. (4.5) and (7.3) do not change β_g at three loops.) Of course it might be the case that Eq. (3.1), and hence Eq. (3.3), is true already in minimal subtraction, in which case no redefinition of g would be required.

We now return to the ideas of coupling constant reduction, which we briefly described earlier. Eqs. (2.14) and (2.16) may be rewritten as

$$Y'^{ijk} \equiv \lambda'^{ijk}(g) = ga^{ijk}, \quad (7.7)$$

where

$$\lambda'^{ijk} = \lambda^{ijk} - g^3 b^{ijk} - g^5 c^{ijk} - \dots \quad (7.8)$$

Eq. (7.7) implies

$$\beta_{Y'}^{ijk} = g^{-1} Y'^{ijk} \beta_g. \quad (7.9)$$

In other words, if the CCR program can be carried out, there is a renormalisation scheme in which the redefined coupling $\frac{Y'}{g}$ is at a fixed point. In our case, we could imagine implementing the CCR program by starting with a solution $Y^{ijk} = ga_1^{ijk}$ of Eq. (2.7) as in Eq. (2.12) and adding corrections to obtain an RG invariant relation as in Eq. (2.16). After redefining λ^{ijk} (and hence Y^{ijk}) as in Eq. (7.8), the new coupling would solve $P = \frac{1}{3}Q$ and would satisfy Eq. (2.10). Hence we would want to find a scheme in which Eq. (2.10) is true upon imposing $P = \frac{1}{3}Q$. Our failure to achieve Eq. (2.19) for a general theory with arbitrary coupling constant redefinitions might already make us suspect that we will also be unable to obtain Eq. (2.10) in general. Let us check this in detail. We readily find from Eqs. (2.2) and (4.9) that the effect of changes δY^{ijk} and δg of the general form Eqs. (4.5) or (7.3) upon the β -function for Y^{ijk} is given in the $P = \frac{1}{3}Q$ case by

$$\delta\beta_Y^{ijk} = 4Qg^2 \delta Y^{ijk} + Y^{m(ij} \delta\gamma^{k)}_m, \quad (7.10)$$

where $\delta\gamma$ is given by Eq. (4.4). Clearly the redefinition we make must at least contain $\delta\gamma_P$ corresponding to substituting Eq. (4.7) in Eq. (4.6). If we make this redefinition alone, we find that β_Y is transformed to

$$\beta_Y'^{ijk} = \beta_Y^{ijk} + \delta\beta_Y^{ijk} = 4Qg^2\delta Y^{ijk} + Y^{m(ij}\gamma'^{k)}_m, \quad (7.11)$$

where

$$\begin{aligned} (16\pi^2)^3\delta Y^{ijk} = & \kappa\left\{\frac{1}{2}g^2S_1({}^i{}_mY^{jk})^m + g^4[C(R)({}^i{}_mC(R)^j{}_nY^k)^{mn} \right. \\ & - \frac{5}{2}Y^{n(jk}C(R)^i){}_mC(R)^m{}_n - \Delta(R)Y^{ijk} \\ & - \frac{1}{2}C(G)C(R)({}^i{}_mY^{jk})^m + 3C(G)^2Y^{ijk}] \\ & \left. + \frac{1}{4}Y^{ilm}Y^{jpq}Y^{krs}Y_{lpr}Y_{mqs}\right\} \end{aligned} \quad (7.12)$$

and

$$\gamma'^i{}_j = \gamma^i{}_j + \delta\gamma^i_{Pj} = \frac{1}{3g}(\beta_g^{(1)} + \beta_g^{(2)} + \beta_g^{(3)})\delta^i{}_j + Qg^2X^i{}_j \quad (7.13)$$

with

$$\begin{aligned} (16\pi^2)^3X = & \kappa\left\{-\frac{4}{3}g^2S_1 \right. \\ & + g^4[4C(R)C(G) + \frac{4}{3}C(R)^2 \\ & \left. + \frac{2}{9}QC(R) + \frac{2}{3}\Delta(R) - 2C(G)^2]\right\} + \frac{1}{27}Q^2. \end{aligned} \quad (7.14)$$

Clearly we can write Eq. (7.11) in the form, correct to three loops,

$$\beta_Y'^{ijk} = g^{-1}\beta_g Z^{ijk}, \quad (7.15)$$

where

$$Z^{ijk} = Y^{ijk} + 4\delta Y^{ijk} + Y^{m(ij}X^k)_m. \quad (7.16)$$

The fact that $Z^{ijk} \neq Y^{ijk}$ means that we have failed to provide an explicit construction of the CCR program in this context. It is not difficult to see that further redefinitions will not save the day. In particular, there is no way to cancel the Y^5 term in δY , which was required to remove the M term in $\gamma^{(3)}$.

Now according to Ref. [14], the existence of a one-loop CCR construction is sufficient to establish it to all orders. If we accept this, then clearly the existence of a_1^{ijk} satisfying Eq. (2.12) assures us that there exists a fixed point of Y^{ijk}/g for a suitably redefined Y . It is disappointing that this result does not emerge naturally in our formalism (in contrast

to the way we found the explicit redefinition that renders a one-loop finite theory three loops finite). At three loops we cannot find a general expression for the required coupling constant redefinition in terms of Y^{ijk} and $C(R)^i_j$ (or equivalently we cannot construct the expansion Eq. (2.16) for a general theory). This does not mean that the redefinition does not exist in special cases; but we must rely on Ref. [14] in order to assert that it is a general result.

8. Conclusions

Our main new result here is the MS expression for the three-loop chiral superfield anomalous dimension in a general $N = 1$ theory, Eq. (5.1). Since in Eq. (3.2) we also gave the three-loop gauge β -function, we have the complete set of β -functions for dimensionless couplings in an arbitrary $N = 1$ theory. It will be interesting to examine the effect of the consequent corrections to the standard running coupling analysis.

Another motivation for the calculation was our previous observation that a certain simple relation between the Yukawa and gauge couplings is RG invariant through two loops. We have found that even with arbitrary coupling constant redefinitions this property does not extend to three loops, in general, at least. That is to say, we are unable to achieve $\gamma^{(3)} \sim \delta^i_j$, so that Eq. (2.19) cannot be true, in general, irrespective of the value of $\beta_g^{(3)}$. We have also shown, however, that given a theory satisfying the (one-loop) $P = \frac{1}{3}Q$ condition, the CCR paradigm assures us of the existence of a fixed point to all orders. This suffices to make such theories phenomenologically interesting.

Acknowledgements

IJ and CGN were supported by PPARC via an Advanced Fellowship and a Graduate Studentship respectively.

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